Separating expansion from contraction in a spherically symmetric universe

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Refs:

MLDM, PRD 81, 123514 (2010) arXiv:0910.5755 [gr-qc]
LDMM, PRD 83: 103528 (2011) arXiv:1103.0976v2[gr-qc]
MLDM, Arxiv: 1302.6186, PRD in press,
LDMMFCA, Arxiv: 1305.3475, PRD in press

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Motivation

- Models of structure formation assume that small local in-homogeneities grow due to gravitational instability, so that the over-densities collapse and eventually form the "bound" structures we observe in the present universe.

- An idea underlying this viewpoint is that the collapse of the over-densities departs from the general expansion of the universe.

- This naturally relates to the interplay between global and local Physics. In particular, local physics seems to be immune to global physics

-On a different context, L. Herrera and co-workers have studied the "cracking" of compact objects in astrophysics using small anisotropic perturbations around spherically symmetric homogeneous fluids in equilibrium.

- The latter references are concerned with the existence of a shell where there is a change in the direction of the radial force acting on the particles of the shells. Whenever this happens one has what they termed as a *cracking* situation, a concept introduced by Herrera in 1992 (PLA 165)

So, we investigate spherically symmetric spacetimes and discuss conditions for the existence of dividing shells separating expanding from collapsing regions within the full GR framework (i.e. non perturbatively).

Outline:

Brief introduction

The coordinate system

Tolman solution for p=0

Local conditions for a separating shell (I) – perfect fluid

Shell crossings

Local conditions for a separating shell (II) – imperfect fluid

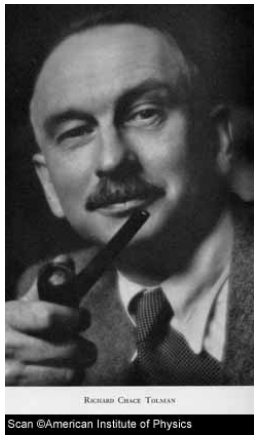
Conclusions

Static (Schwarzschild 1916, de Sitter 1917)

Spherically symmetric Models

Expanding (Friedmann 1921, Lemaître 1926-1933 Tolman 1934,…)

Contracting (Lemaître 1933, Tolman 1934, Bondi 1947, Chandrasekhar, 1930,…)



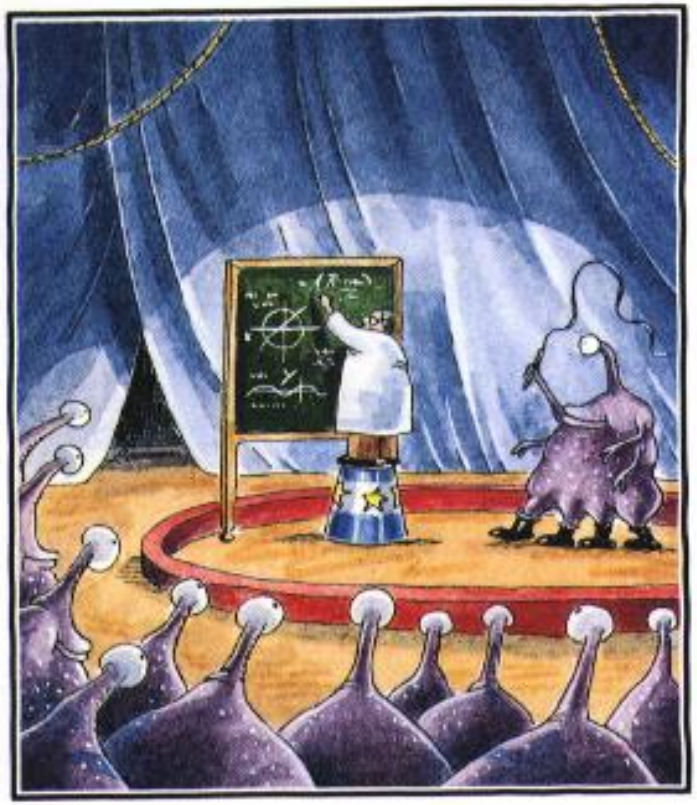
We aim at analysing the interplay of the two main regimes that where so intensively studied in a separate way, i.e., spherical collapse and expansion within GR

Similar concerns have been pursued in several works in the literature:

H. Bondi, MNRAS 142 (1969) 142); (bounces)
W. B. Bonnor, Mon. Not. R. Astron. Soc. 167, 55 (1974)
J. Barrow, G Galloway, and F.Tipler, MNRAS 223 (1986) (recollapse)
B. Carr, A. Coley, Phys.Rev. D62 (2000) 044023

How local physics departs and becomes immune from the global expansion.

A.Einstein and E.G. Strauss, Rev.Mod. Phys. 17,120 (1945),
ibid 18,148 (1946)
G. F. R. Ellis, Local and Global Physics, Int. J. Mod. Phys A17, 2667 (2002)
V. Faraoni and A. Jacques, Phys. Rev. D 76, 063510 (2007),



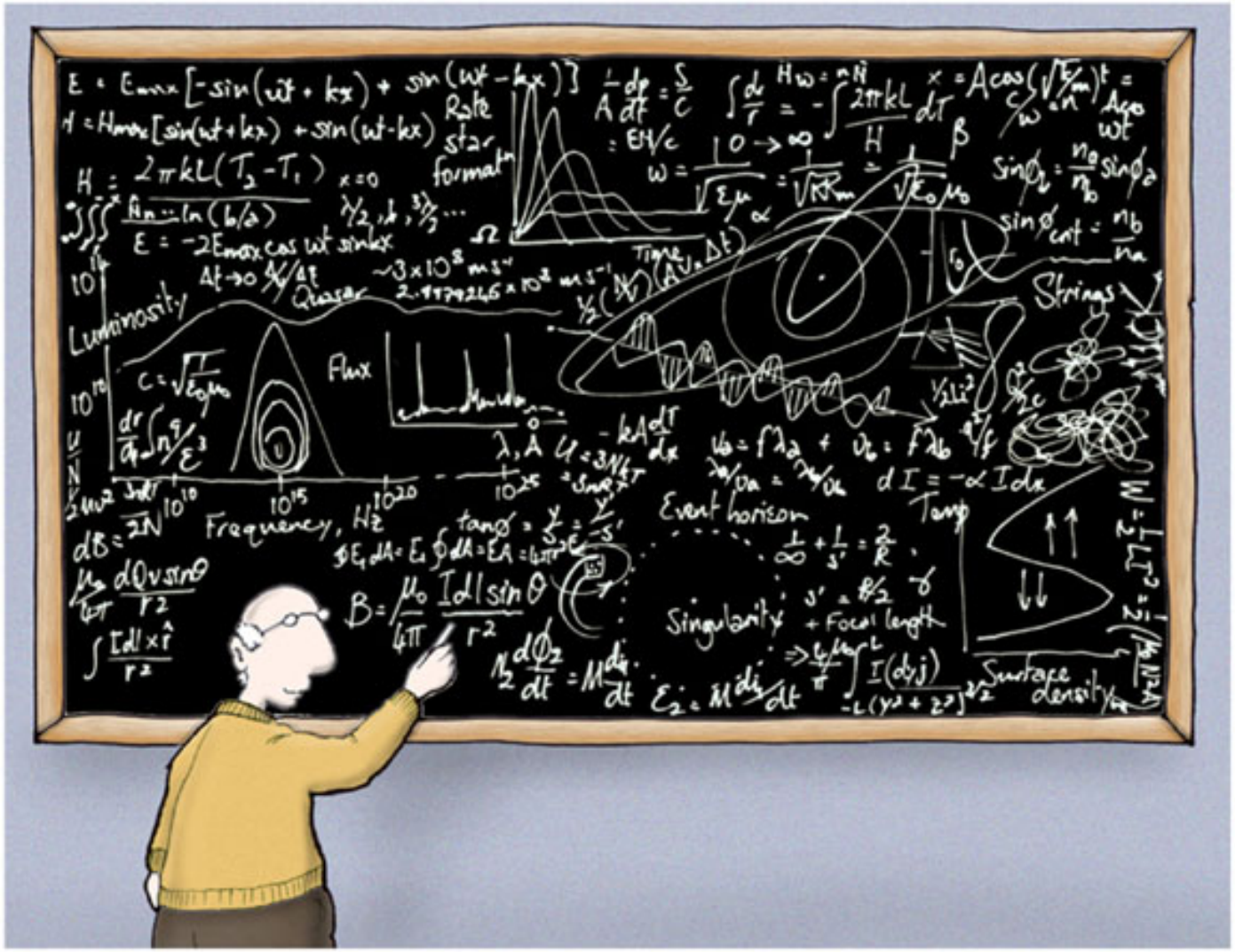
Abducted by an alien circus company, Professor Doyle is forced to write calculus equations in center ring.

(1939) R. Oppenheimer and G. Volkoff derived a condition for the equilibrium of a spherical configuration such as a (neutron) star. It became known as the TOV equation of state

$$\frac{dP}{dr} = -\frac{(\rho + p)(M/r^2 + 4\pi \mathrm{Pr})}{1 - 2M/r}$$

In 1972 Gunn and Gott put foward their influential model for spherical collapse model.

[more recently...Manera & Mota, 2006, Nunes & Mota 2006, Pace et al 2010...]



Astrophysics made simple

Familiar LTB form of Einstein equations:

$$ds^2 = -\alpha(T,R)^2(\partial_T t)^2 dT^2 + \frac{(\partial_R r)^2}{1+E(T,R)} dR^2 + r^2 d\Omega^2,$$

$$\beta = -\dot{r}.$$

$$\dot{r}^2 = \alpha^2\left(2\frac{M}{r} + \frac{1}{3}\Lambda r^2 + E\right)$$

The LTB solutions

$$p = 0$$

$$M = M(R)$$

$$E = E(R)$$

$$d\eta = \sqrt{E}\,\frac{dt}{r}$$

$$h(\eta) = \eta - \sin\eta \quad if \quad E < 0$$

$$h(\eta) = \sinh\eta - \eta \quad if \quad E > 0$$

$$t - t_{BB}(R) = \frac{M(R)}{2(\sqrt{E}\,)^3}\,h(\eta)$$

3+1 Splitting

$$N^a_b := -n^a n_b, \qquad h^{ab} := g^{ab} + n^a n^b,$$

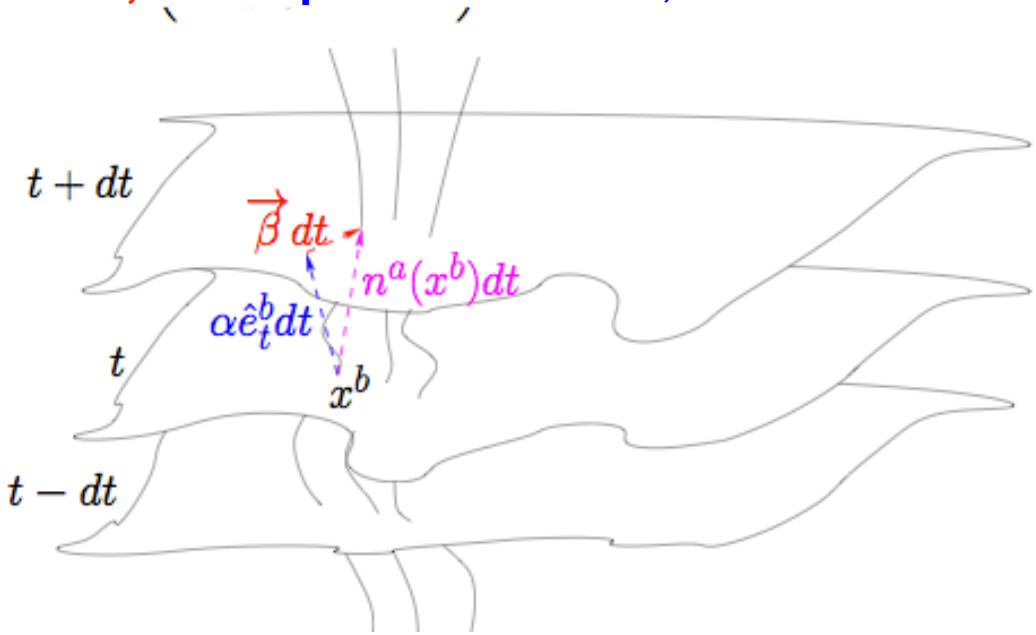
$$n_a := -\alpha \nabla_a t = [-\alpha, 0, 0, 0] \, (n_a n^a = -1)$$

Use **Generalised Painlevé-Gullstrand coords (also, Gautreau)**

[Lasky and Lun PRD 2006, 2007]

$$ds^2 = -\alpha^2 dt^2 + \frac{\left(dR + \beta(t,R)\, dt\right)^2}{1 + E(t,R)} + r^2(t,R)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

α= lapse function, **β=shift function,** **E =curvature-energy**



Do 3+1 decomposition

$$n_{a;b} = N^c_b n_{a;c} + n_{\bar{a};\bar{b}} = -n_b \dot{n}_a + \tfrac{1}{3}\,\Theta h_{ab} + \sigma_{ab} + \omega_{ab},$$

$$T^{ab} = \rho\, n^a n^b + p\, h^{ab} + \Pi^{ab} + 2\, j^{(a} n^{b)}$$

$$\sigma_{ab} = \sigma(t,R)\, P_{ab}$$

$$P_i^{\,j} = diag(-2,1,1)$$

$$E_{ab} = \Sigma(t,R)\, P_{ab}$$

$$\Pi_{ab} = \Pi(t,R)\, P_{ab}$$

$$\frac{1}{\alpha}\left(D_a D_b - \frac{1}{3}\,{}^3g_{ab} D^c D_c\right)\alpha = \varepsilon(t,R)\, P_{ab}$$

$$\theta = n^a{}_{;a}$$

$${}^3R_{ab} - \frac{1}{3}\,{}^3R\;{}^3g_{ab} = q(t,R)\, P_{ab}$$

Introducing the Misner-Sharpe (also ADM) mass

$$M = \int \rho r^2 dr$$

And restricting to a perfect fluid

$$\dot{M} = \beta 4\pi P r^2 = 4\pi P r^2 \alpha \sqrt{2\frac{M}{r} + \frac{1}{3}\Lambda r^2 + E},$$

$$\dot{E} r' = 2\beta \frac{1+E}{\rho + P} P' = 2\frac{1+E}{\rho+P} P' \alpha \sqrt{2\frac{M}{r} + \frac{1}{3}\Lambda r^2 + E}$$

Bianchi contracted identities $\quad T^a_{b;a} = 0$

$$n^b T^a_{b;a} = -\mathcal{L}_n \rho - (\rho + P)\Theta = 0,$$

$$\Rightarrow P' = -(\rho + P)\frac{\alpha'}{\alpha}.$$

Local conditions for a dividing shell:

$$0 = r\left(\frac{\Theta}{3} + a\right) = -\frac{\beta}{\alpha} = \mathcal{L}_n r, \quad \Rightarrow \quad E_\star = -2\frac{M_\star}{r_\star},$$

$$0 = -r\left[\mathcal{L}_n\left(\frac{\Theta}{3} + a\right) - \left(\frac{\Theta}{3} + a\right)^2\right]$$

$$= -\mathcal{L}_n^2 r.$$

$$\Rightarrow \quad \mathrm{gTOV} = \left[\frac{1+E}{\rho+P}P' + 4\pi P r + \frac{M}{r^2}\right]. = 0$$

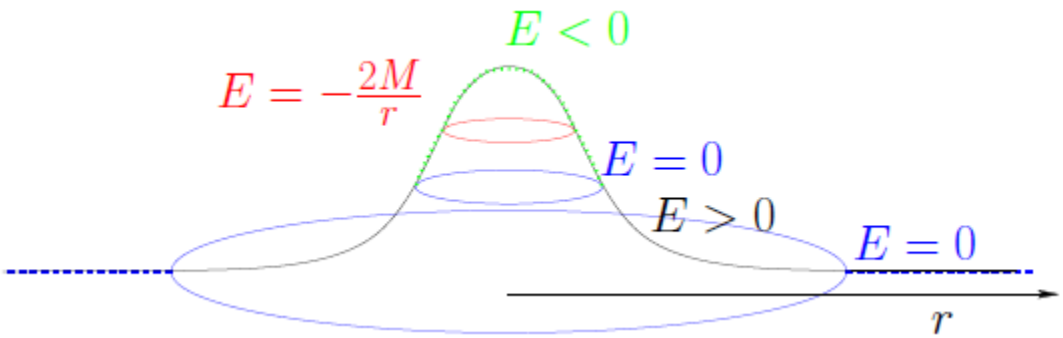
$$\Theta_\star = 0 \quad \Rightarrow \quad \left(\frac{\beta}{\alpha}\right)'_\star = 0, \quad \Rightarrow \quad \text{a (shear) } = 0$$

NB: E is not 3-curvature!

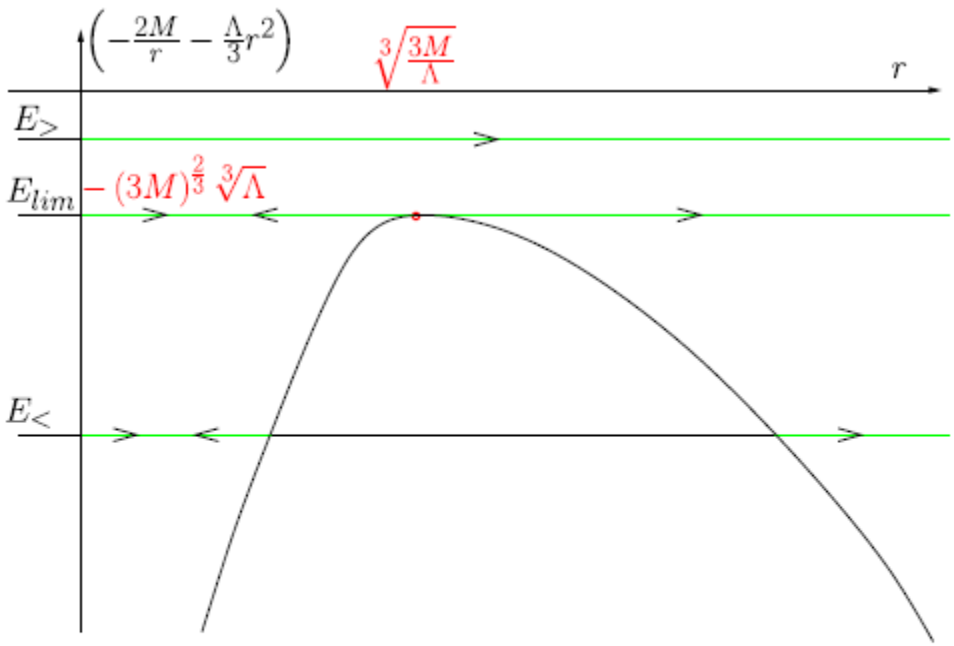
$$-\frac{{}^3R}{2} = \frac{1}{r^2}\left[(1+E)\,r'^2 + E'r'r + 2\,(1+E)\,r''r - 1\right] \ ,$$

But the turning point requires positive curvature

$${}^3R + \tfrac{2}{3}\Theta^2 - 6a^2 = 16\pi\rho + 2\Lambda,$$



“Illustration” :: Λ-CDM



$$r_{lim} = \sqrt[3]{\frac{3M}{\Lambda}},$$

$$E_{lim} = -(3M)^{\frac{2}{3}}\Lambda^{\frac{1}{3}} = -\Lambda r_{lim}^2,$$

FIG. 1 (color online). Kinematic analysis for a given shell of constant M and E. Depending on E relative to $E_{\rm lim}$, the fate of the shell is either to remain bound ($E_< < E_{\rm lim}$) or to escape and cosmologically expand ($E_> > E_{\rm lim}$). There exists a critical behavior where the shell will forever expand, but within a finite, bound radius ($E = E_{\rm lim}$, $r \leq r_{\rm lim}$). The maximum occurs at $r_{\rm lim} = \sqrt[3]{3M/\Lambda}$.

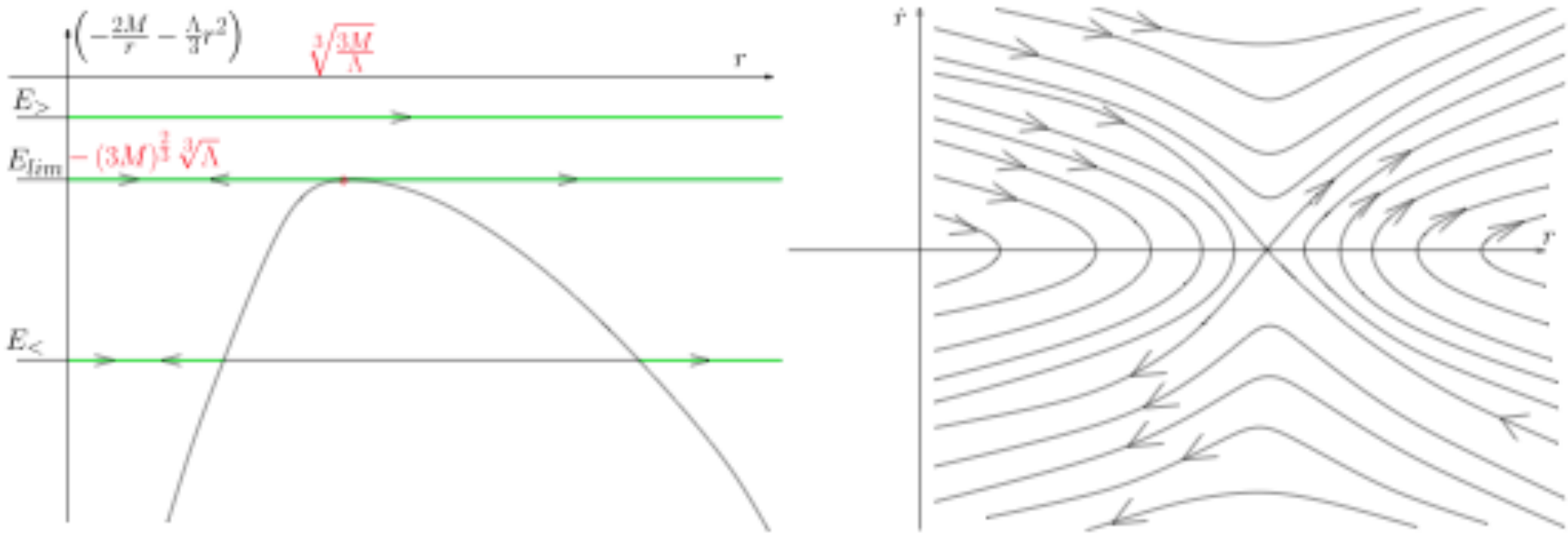


FIG. 1 (color online). Effective potential kinematic analysis (left) and phase space analysis (right) from [5]. The kinematic analysis for a given shell of constant M and E depict the fate of the shell, depending on E relative to $E_{\rm lim}$. It either remains bound ($E_< < E_{\rm lim}$) or escapes and cosmologically expands ($E_> > E_{\rm lim}$). There exists a critical behavior where the shell will forever expand, but within a finite, bound radius ($E = E_{\rm lim}$, $r \leq r_{\rm lim}$). The maximum occurs at $r_{\rm lim} = \sqrt{3M/\Lambda}$. The corresponding phase space behavior follows, the scales are set by the value of $r_{\rm lim} = \sqrt{3M/\Lambda}$ while the actual kinematic of the shell is given by E.

Models with anisotropic pressures (but without heat fluxes)

$$T_{\mu\nu} = (\rho + P)n_\mu n_\nu + P g_{\mu\nu} + 2j_{(\mu}n_{\nu)} + \Pi_{\mu\nu}.$$

$$\Pi_{ij} := \Pi(t, r)P_{ij}.$$

$$K_{ij} - \tfrac{1}{3}\perp_{ij} K := a(t, r)P_{ij}.$$

$$^3R_{ij} - \tfrac{1}{3}\perp_{ij} {}^3R := q(t, r)P_{ij}.$$

Eij = (t; r) Pij

$$R = r \Longrightarrow \partial R / \partial r = 1$$

$$(\mathcal{L}_n R)^2 = \frac{2M}{R} + (1 + E)\left(\frac{\partial R}{\partial r}\right)^2 - 1.$$

$$\frac{M}{R^2} + 4\pi(P - 2\Pi)R = \frac{1 + E}{\alpha}\frac{\partial \alpha}{\partial r}\frac{\partial R}{\partial r} - \mathcal{L}_n{}^2 R,$$

$$\mathcal{L}_n M = -4\pi R^2\left[(P - 2\Pi)\mathcal{L}_n R + j\frac{\partial R}{\partial r}(1 + E)\right]$$

$$
\begin{aligned}
{}^3R + \frac{2}{3}\theta^2 - 6a^2 =& 16\pi\rho + 2\Lambda \\
2\mathcal{L}_n\theta - \frac{1}{2}{}^3R - \theta^2 - 9a^2 + \frac{2}{\alpha}D^aD_a\alpha =& 24\pi P - 3\Lambda, \\
-\mathcal{L}_n a + a\theta + \epsilon - q =& 8\pi\,\Pi.
\end{aligned}
$$

$$
(\mathcal{L}_n r)^2 = \frac{2M}{r} + E = 0
$$

$$
-\mathcal{L}_n^2\, r = \frac{M}{r^2} + 4\pi(P - 2\Pi)r - \frac{1+E}{\alpha}\frac{\partial\alpha}{\partial r} = 0\ .
$$

$$
-\frac{1}{\alpha}\frac{\partial\alpha}{\partial r} = \frac{1}{(\rho + P - 2\Pi)}\left[\frac{\partial}{\partial r}(P - 2\Pi) - \frac{6\Pi}{r}\right]
$$

$$L_n r = 0 = \frac{2M}{r} + E$$

$$gTOV = L_n{}^2 r = 0 = \frac{M}{r^2} + 4\pi(P - 2\Pi)r - \frac{(1+E)}{\alpha}\frac{\partial\alpha}{\partial r}$$

An illustration can be extrapolated from the solution by R.Sussman and D. Pavón, PRD 60, 104023 (1999) generalising it to the cases where E is not vanishing

$$\mathcal{L}_n a - aK + \epsilon - q = 8\pi\Pi.$$

$$\epsilon - q = 8\pi\,\Pi = 2\Sigma$$

[Mimoso & Crawford, CQG 1993, Coley & Mac Manus, CQG 1994]

$$-2\mathcal{L}_n\Theta = \frac{{}^3R}{2} + \Theta^2 + 9a^2 - \frac{2}{\alpha}D^\mu D_\mu \alpha + 24\pi P - 3\Lambda,$$

$$\mathcal{L}_n a = -a\Theta + \epsilon - q + 8\pi\,\Pi$$

$$\mathcal{L}_n\Sigma = -4\pi\mathcal{L}_n\Pi - 4\pi\left(\rho + P - 2\Pi\right)a$$

$$-\left(3\Sigma + 4\pi\Pi\right)\left(\frac{\Theta}{3} + a\right),$$

$$\left(\frac{\Theta}{3} + a\right)' = -3a\frac{r'}{r},$$

$$\frac{4\pi}{3}\left(\rho + 3\Pi\right)' = -\Sigma' - 3\left(\Sigma + 4\pi\Pi\right)\frac{r'}{r},$$

$${}^3R + \frac{2}{3}\Theta^2 - 6a^2 = 16\pi\rho + 2\Lambda,$$

$$\mathcal{L}_n\rho = -\Theta(\rho + P) - 6\Pi\,a$$

$$0 = \left(D^k + \dot{n}^k\right)\left(\Pi_{ik} + h_{ik}P\right) + \left[\rho - \left(P - 2\Pi\right)\right]\dot{n}_i$$

$$-n_i\left[\Theta P + 6\Pi a\right].$$

Introducing the Misner-Sharp mass [18] and following [9]

$$M' = 4\pi\rho r^2 r' \tag{I.22}$$

it is possible to derive[3]

$$(\mathcal{L}_n r)^2 = \frac{2M}{r} + (1+E)\,(r')^2 - 1 + \frac{1}{3}\Lambda r^2 \tag{I.25}$$

and

$$-\mathcal{L}_n^2\, r = \frac{M}{r^2} + 4\pi(P - 2\Pi)r - (1+E)\,\frac{\alpha'}{\alpha}r' - \frac{1}{3}\Lambda r\,. \tag{I.26}$$

This allows us to extend the generalization of the TOV function made in [1] to the case where anisotropic stresses are present:

$$\mathrm{gTOV} = -\mathcal{L}_n^2\, r\,. \tag{I.27}$$

$$-\frac{\alpha'}{\alpha} = \frac{1}{(\rho + P - 2\Pi)}\left[(P - 2\Pi)' - 6\Pi\frac{r'}{r}\right]$$

$$\mathrm{gTOV} = -\mathcal{L}_n^2\, r = \frac{M}{r^2} + 4\pi(P - 2\Pi)r + \frac{(1+E)\,r'}{(\rho + P - 2\Pi)}\left[(P - 2\Pi)' - 6\Pi\frac{r'}{r}\right] - \frac{1}{3}\Lambda r$$

Sussman and Pavon PRD 1999

$$M_{dust} = M(R)$$

$$M_{rad} = \frac{W(R) r_i(R)}{2r(T,R)}$$

$$(P - 2\Pi)' - 6\Pi \frac{r'}{r} = 0$$

$$\dot{r}^2 = c^2 \left(2\frac{M}{r} + \frac{W r_i}{r^2} + E \right)$$

$$\frac{\ddot{r}}{c^2} = -\frac{M}{r^2} - \frac{W r_i}{r^3},$$

$$\frac{W r_i}{2r^3} = 4\pi \left(P - 2\Pi\right) r.$$

$$r_\star = \sqrt[3]{\frac{M_\star}{8\pi \left(2\Pi_\star - P_\star\right)}}$$

$$W_\star = -M_\star \quad \Rightarrow W(R_\star) < 0 \quad \Leftrightarrow P_\star < 2\Pi_\star.$$

$$E_\star = -\frac{M_\star}{r_\star} = -\sqrt[3]{8\pi \left(2\Pi_\star - P_\star\right)} M_\star^{\frac{2}{3}} = \frac{M_\star^{\frac{2}{3}} W_\star^{\frac{1}{3}}}{r_\star} < 0.$$

From local to global?

Shell crossing in initially expanding Λ-CDM

Assumptions:

(1) Regular density distribution (no vacuum at the center, finiteness of mass)

(2) Initial Hubble-type flow >> $E<E_{lim}$ at center

(3) Asymptotic spatial cosmological behaviour (--> FLRW)

Non-locality:

$$r = r_\star,$$

$$\left(\frac{\beta}{\alpha}\right) = -r\left(a + \frac{\Theta}{3}\right) = \left(\frac{\beta}{\alpha}\right)_{r_0}\left(\frac{r_0}{r}\right)^2 - \frac{1}{r^2}\int_{r_0}^{r}\Theta r^2 dr,$$

$$\left(\frac{\beta}{\alpha}\right)_{r_0} r_0^2 = \int_{r_0}^{r_\star}\Theta r^2 dr,$$

$$\left(a + \frac{\Theta}{3}\right) = \left(\frac{r_0}{r}\right)^3\left(a_{r_0} + \frac{\Theta_{r_0}}{3}\right) + \frac{1}{r^3}\int_{r_0}^{r}\Theta r^2 dr,$$

$$\mathcal{L}_n\left(\frac{\beta}{\alpha}\right) = \left(\frac{\beta}{\alpha}\right)\left[\frac{2}{r^3}\left(r_0^2\left(\frac{\beta}{\alpha}\right)_{r_0} - \int_{r_0}^{r}\Theta r^2 dr\right) + \Theta\right]$$

$$+ \frac{1}{\alpha r^2}\left[r_0^2\partial_t\left(\frac{\beta}{\alpha}\right)_{r_0} - \int_{r_0}^{r}\dot{\Theta} r^2 dr\right] = \mathrm{gTOV}.$$

$$\mathrm{gTOV} = 0,$$

$$\left(\frac{\beta}{\alpha}\right)\left[\frac{2}{r^3}\left(r_0^2\left(\frac{\beta}{\alpha}\right)_{r_0} - \int_{r_0}^{r}\Theta r^2 dr\right) + \Theta\right]$$

$$= \frac{1}{\alpha r^2}\left[r_0^2\partial_t\left(\frac{\beta}{\alpha}\right)_{r_0} - \int_{r_0}^{r}\dot{\Theta} r^2 dr\right].$$

$$r_0^2\partial_t\left(\frac{\beta}{\alpha}\right)_{r_0} = \int_{r_0}^{r_\star}\dot{\Theta} r^2 dr,$$

"Cracking" (Herrera 1992)

L. Herrera, Phys. Lett. A **165**, 206 (1992). A. Di Prisco, L. Herrera, E. Fuenmayor, and V. Varela, Phys. Lett. A **195**, 23 (1994).

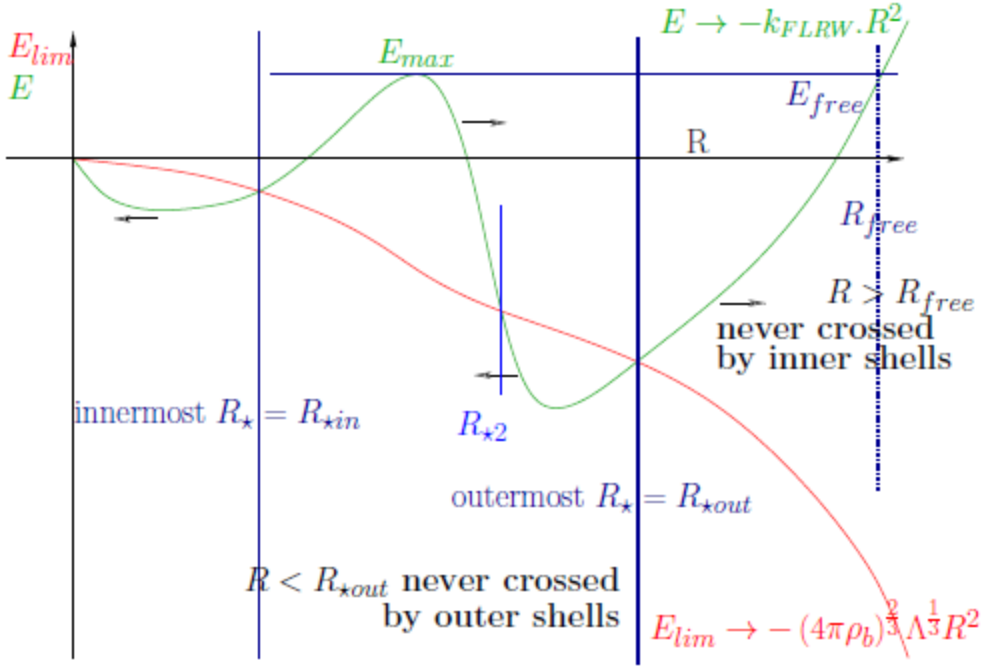


Figure 8. Open background with arbitrary central mass distribution and a single local undercoming intersection. It always gives protected inner shells as well as unmodified cosmological expansion, when keeping integrability despite shell-crossing. Shell crossing entails no fundamental modification.

Conclusions

We have found 2 local conditions for the existence shells separating an inner collapsing region from an outer expansion.

(i) A particular balance between the so-called energy function of the model and the potential energy (Existence of turning point).

(ii) A stationarity condition which demands that a generalization of the Tolman-Oppenheimer-Volkov equilibrium condition be satisfied by the separating shell.

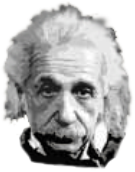
In an attempt to go beyond local conditions we have analysed the implications of shell crossings on a cosmologically motivated model.

The cosmologically motivated problem requires the consideration of a content with anisotropic stresses and, most likely, energy transfer as well.

THANKS for listening !

Acknowledgments:
Finantial support of grants PTDC/FIS/102742/2008;
CERN/FP/116398/2010 :

http://www.idpasc.lip.pt/LIP/events/2013_modified_gravity_theories/index.php



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Modified Gravity Theories: Beyond Einstein's Legacy

Gravity Theories extending General Relativity have attracted a great deal of interest since the advent of Einstein's theory. The first attempts sought to unify Gravitation and Electromagnetism and were followed by the search for a grand unified theory that would also merge the weak and strong nuclear forces. More recently, the previous endeavor has been augmented with the possibility that Modified Gravity might explain the current accelerated expansion of the Universe.

Indeed, relatively simple extensions to Einstein's General Relativity can deliver an accelerated Universe without resorting to the inclusion of exotic matter such as the cosmological constant or, in general terms, dark energy.

Numerous studies have been devoted to the investigation of issues such as extra-dimensions, variations of constants, ...

The meeting stems from an International Doctorate Network in Particle Physics, Astrophysics and Cosmology -- IDPASC and aims at

1. Training Phd Students and Young Researchers
2. Setting an appropriate atmosphere for an open and constructive discussion on topical areas of research beyond Einstein's legacy.

In this framework, the meeting will consist of a mix of invited lectures presented by specialists in the various fields, supplemented by tutorials, and short communications delivered by the young researchers and students.

It will take place at the campus of the Faculty of Science of the University of Lisbon on the 16th, 17th and 18th of September of 2013. (see Venue)

16, 17 E 18 OF SEPTEMBER 2013

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Invited Lecturers

(I) Prof. John Parsons (Universidade de Columbia, EUA)
"TBA/CERN/LHC-ATLAS-CMS related"

(II) Prof. Jan Hamann (CERN, Planck)
"The Cosmic Microwave Background and Results from Planck"

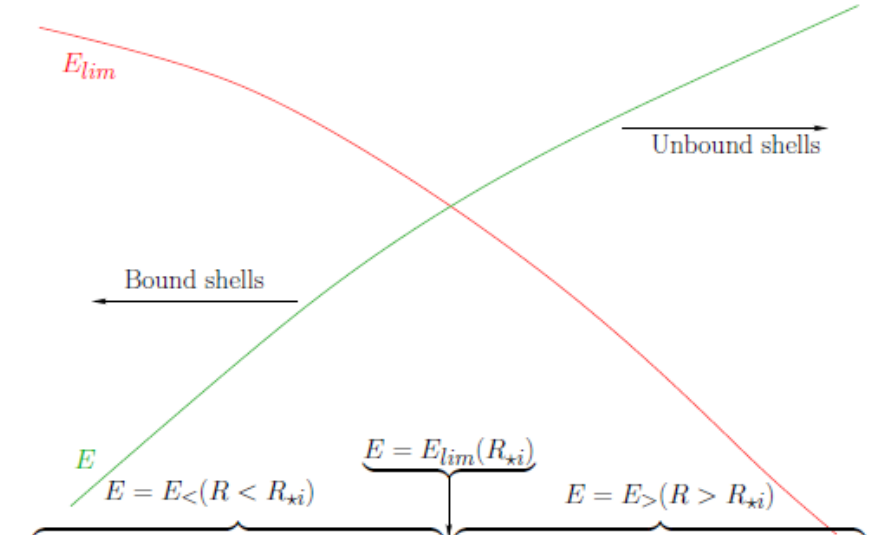
(III) Prof. Paulo Freire (Max Planck Institute for Radio Astronomy)
"Pulsars, binary pulsars, pulsar timing and historical tests of general relativity"
"Current pulsar tests of general relativity and alternative theories of gravity"

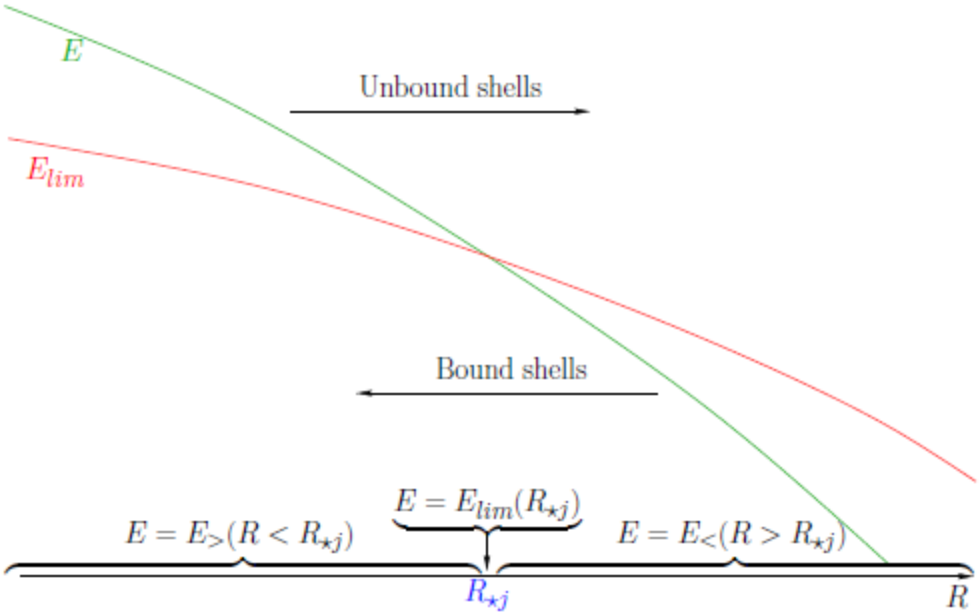
(IV) Prof. Francisco Lobo (CAAUL, U. Lisboa)
"Extended Theories of Gravity and the Acceleration of the Universe"

(5) Local mass of crossing shell is conserved

$$r_{lim} = \sqrt[3]{\frac{3M}{\Lambda}},$$

$$E_{lim} = -\left(3M\right)^{\frac{2}{3}} \Lambda^{\frac{1}{3}} = -\Lambda r_{lim}^2,$$





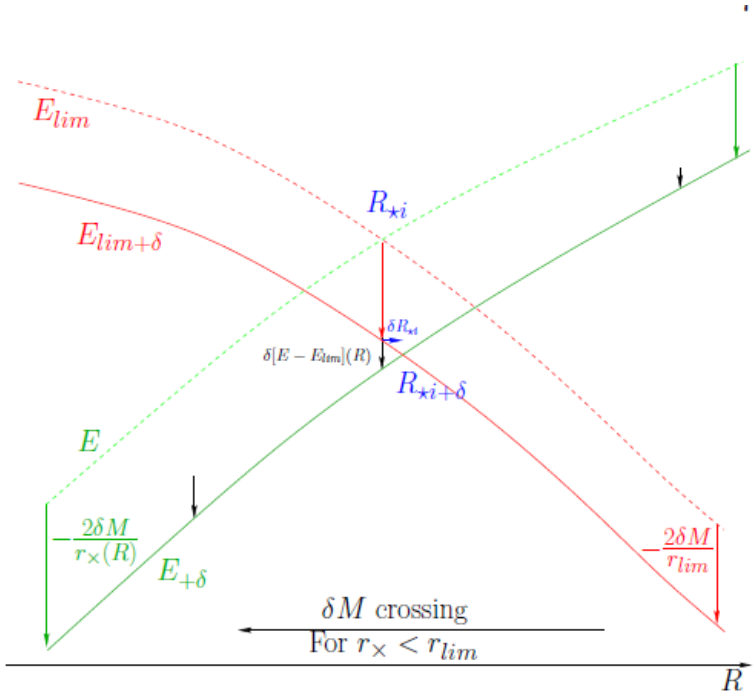


Figure 4. Effect of an ingoing, infinitesimal test shell-crossing on the energy and critical energy profiles, around the *local* initial configuration for the overcoming of E_{lim} by E. The initial intersection shell becomes bound on such perturbations and the local intersection shell shifts outwards in radius.

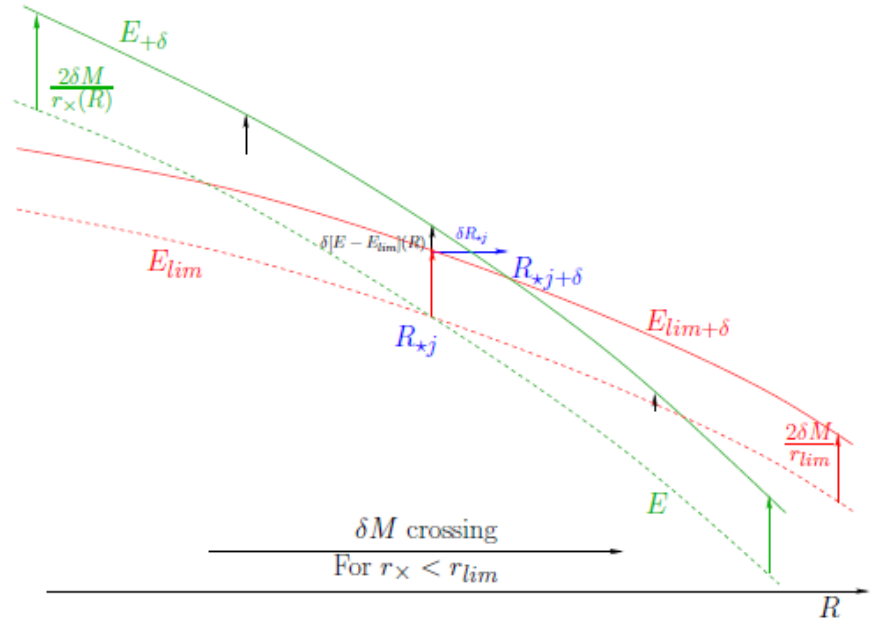


Figure 5. Effect of an outgoing, infinitesimal shell-crossing on the energy and critical energy profiles, around the *local* initial configuration for the undercoming of E_{lim} by E. The initial intersection shell becomes unbound on such perturbations and the local intersection shell shifts outwards in radius.

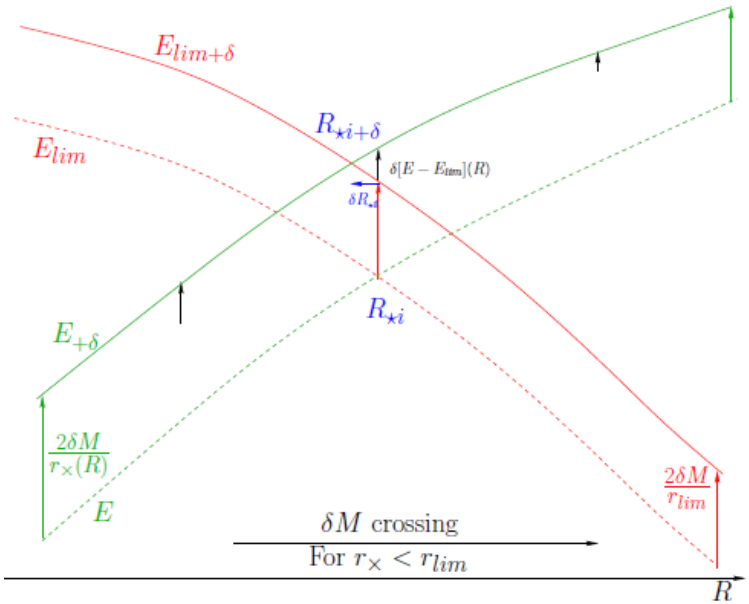


Figure 6. Effect of an outgoing, infinitesimal shell-crossing on the energy and critical energy profiles, around the *local* initial configuration for the overcoming of E_{lim} by E. The initial intersection shell becomes unbound on such perturbations and the local intersection shell shifts inwards in radius.

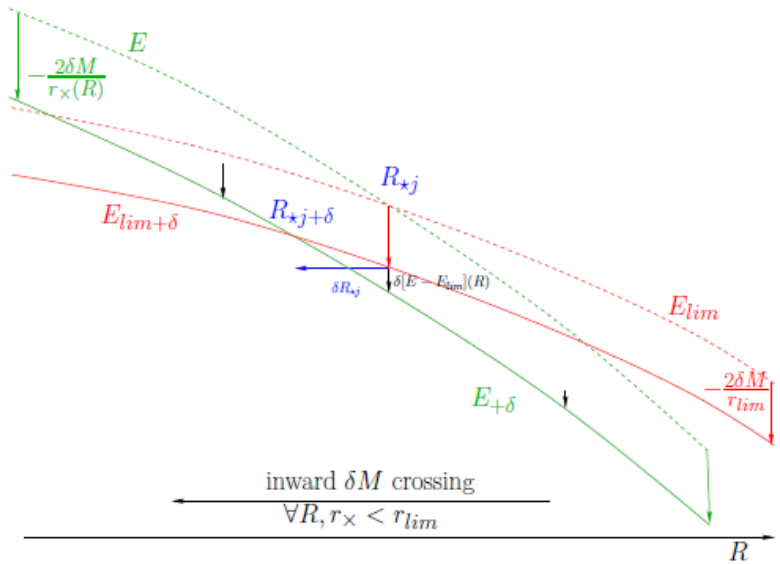


Figure 7. Effect of an ingoing, infinitesimal test shell-crossing on the energy and critical energy profiles, around the *local* initial configuration for the undercoming of E_{lim} by E. The initial intersection shell becomes bound on such perturbations and the intersection shell shifts inwards in radius.

Dictionary

$$\left(\frac{\Theta}{3}+a\right)=\frac{\mathcal{L}_n r}{r}$$

$$\frac{2M}{r^3}+(1+E)\left(\frac{r'}{r}\right)^2-\frac{1}{r^2}=\frac{8\pi\rho}{3}-\frac{{}^3R}{2}+2a\left(\frac{\Theta}{3}+a\right)$$

$$\frac{{}^3R}{3}+\frac{8\pi\rho}{3}+2a\left(\frac{\Theta}{3}+a\right)$$
$$=\frac{2M}{r^3}+2\frac{\sqrt{1+E}}{r}\left(\sqrt{1+E}r'\right)$$

$$q=\frac{1}{6}\left\{r\left(\frac{Er'}{r^2}\right)'+E\frac{r''}{r}+\frac{2}{r^2}\left[1+r^2\left(\frac{r'}{r}\right)'\right]\right\}$$
$$=\frac{1}{6}\left[r\left(\frac{Er'}{r^2}\right)'+(2+E)\frac{r''}{r}+\frac{2}{r^2}\left(1-r'^2\right)\right],$$

$$\frac{1}{\alpha}D^\mu D_\mu\alpha=\frac{\sqrt{1+E}}{\alpha r^2}\left(r^2\sqrt{1+E}\alpha'\right)',$$
$$\epsilon=-\frac{r\sqrt{1+E}}{3\alpha}\left(\frac{\sqrt{1+E}}{r}\alpha'\right)'$$

Cracking

$$\mathcal{L}_n\left(\frac{\Theta}{3}+a\right) = \epsilon + \frac{1}{3\alpha}D^k D_k \alpha - \left(\frac{\Theta}{3}+a\right)^2 - \left\{\Sigma + \frac{4\pi}{3}\left[\rho + 3\left(P-\Pi\right)\right]\right\} + \frac{\Lambda}{3}, \tag{2.19}$$

$$\mathcal{L}_n a = -\frac{2}{3}a\Theta + a^2 + \epsilon - \left(\Sigma - 4\pi\Pi\right), \tag{2.20}$$

$$\mathcal{L}_n \Sigma = -4\pi\left\{\mathcal{L}_n \Pi + a\left(\rho + P - 2\Pi\right)\right\} - \left(3\Sigma + 4\pi\Pi\right)\left(\frac{\Theta}{3}+a\right), \tag{2.21}$$

$$\Sigma + 4\pi\Pi = q + a\left(\frac{\Theta}{3}+a\right).$$

$$\left(\frac{\Theta}{3}+a\right)^2 = \frac{8\pi\rho}{3} - \frac{{}^3R}{6} + \frac{\Lambda}{3} + 2a\left(\frac{\Theta}{3}+a\right)$$

$$\left(P-2\Pi\right)' = 6\Pi\frac{r'}{r} - \left(\rho + P - 2\Pi\right)\frac{\alpha'}{\alpha}, \tag{2.24}$$

$$\left(\frac{\Theta}{3}+a\right)' = -3a\frac{r'}{r}, \tag{2.25}$$

$$\frac{4\pi}{3}\rho' + \frac{\left(\left(\Sigma + 4\pi\Pi\right)r^3\right)'}{r^3} = 0\,. \tag{2.26}$$

$$R_{ab} - \frac{1}{2} g_{ab} R - \Lambda g_{ab} = 8\pi G T_{ab}$$